Chapter 9.

Bernoulli and Binomial distributions.

Situation

It is April 1st and Ms Hamish, a teacher of mathematics, decides to see if she can trick her year 12 class into thinking they have a surprise test.
She walks into class brandishing some sheets of paper and announces:

"Okay separate the desks we have a spot test on probability today."
"Twenty questions, multiple choice, no talking"

The surprised, and not a little alarmed, students express their concerns:

"You didn't tell us we had a test." *"Can we use our calculators?"*
"Does the mark count towards our assessment?"
"Are notes allowed?" *"That's not fair."* *"Does it count?"*
"Are other classes doing the same test?"
"Come on Miss, give us a break." *"What did you say it was on?"*
"Is this an April Fool joke Miss?"

Ignoring all such comments Ms Hamish walks around the class giving out a response sheet to each student. Each sheet shows the numbers 1 to 20 each with five possible responses (a) to (e). Part of such a sheet is shown below:

Probability test. Name: ______________________

For each question circle one response out of (a), (b), (c), (d) or (e).

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	

"You have one minute to make your twenty choices." says Ms Hamish.
"Start now."
"This is an April Fool trick isn't it Miss? asks one student.
"Well yes it is actually, and there are no questions to go with the responses, but I want you all to do it anyway because we will discuss the responses later."

At the end of the minute all students have made their twenty random guesses, as has Ms Hamish on a sheet which she then proclaims as "the answer sheet"!
As she reads out the "correct" responses according to the random guesses she made the students mark their response sheets.

Given that there were 21 students in the class and each student could have a final score of 0, 1, 2, 3, ... 20 roughly how many students would you expect to get each score? Make a table of your estimates and then compare and discuss your table with others in your class.

Bernoulli distributions.

Considering the situation from the previous page, with the possible final scores being the discrete values 0, 1, 2, 3, ..., 20, and there being 21 students in the class would we expect the following uniform distribution?

Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Number of students.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Such a distribution would be somewhat unlikely, to say the least.

If we consider the situation of the previous page to involve 20 *trials*, with each trial being that of answering a question, then for each trial there are two possibilities, which we will call success and failure. If we call the failure a zero and the success a 1, we have a discrete random variable with the two possible values, 0 and 1, with probabilities 0·8 and 0·2 respectively.

x	0	1
$P(X = x)$	0·8	0·2

A trial which can be considered to have just two mutually exclusive and exhaustive outcomes, sometimes referred to as *success* and *failure*, is called a **Bernoulli trial**, named after one of the famous Bernoulli family of Swiss mathematicians. The associated random variable, with its two possible values, 0 and 1, is called a **Bernoulli random variable**. If the probability of success is p then the probability of failure will be (1 – p).

	Failure	Success
x	0	1
$P(X = x)$	1 – p	p

We say that the Bernoulli random variable has **parameter** p, the probability of obtaining a 1. The parameter is a constant *characteristic* of the situation. Each time the Bernoulli trial is carried out the probability of success is p. Knowing p allows us to determine the probabilities associated with each value of the Bernoulli random variable.

If we roll a normal six sided die there are six possible outcomes, 1, 2, 3, 4, 5 and 6. We have a uniform discrete random variable with probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

However, if we are only concerned with "getting a 6", which we will call a success, or "not getting a 6", which we will call a failure, we have a Bernoulli random variable, with probability distribution:

	Failure	Success
x	0	1
$P(X = x)$	$\frac{5}{6}$	$\frac{1}{6}$

Some examples of Bernoulli random variables:

- Flipping a coin with obtaining a head being considered a success.

	Tail	Head
x	0	1
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

- Guessing the answer to a multiple choice situation in which there are four answers to choose from, only one of which is correct.

	Wrong answer	Correct answer
x	0	1
$P(X = x)$	$\frac{3}{4}$	$\frac{1}{4}$

- A randomly selected seed germinating (success) or not.

	Not germinating	Germinating
x	0	1
$P(X = x)$	$1 - p$	p

The value of p being estimated by experiment.

Applying $\quad E(X) = \sum(x_i\, p_i)$,

and $\quad Var(X) = \sum[p_i\,(x_i - E(X))^2] \quad$ or $\quad Var(X) = E(X^2) - [E(X)]^2$

to the **Bernoulli distribution** with parameter p:

x	0	1
$P(X = x)$	$1 - p$	p

$$\begin{aligned} E(X) &= 0 \times (1 - p) + 1 \times p \\ &= p \\ Var(X) &= (1 - p) \times (0 - p)^2 + p \times (1 - p)^2 \\ &= p(1 - p)(p + 1 - p) \\ &= p(1 - p) \end{aligned}$$

☞ The long term mean, or expected value $E(X)$, of a Bernoulli distribution with parameter p is p.

☞ The variance of a Bernoulli distribution with parameter p is $p(1 - p)$.

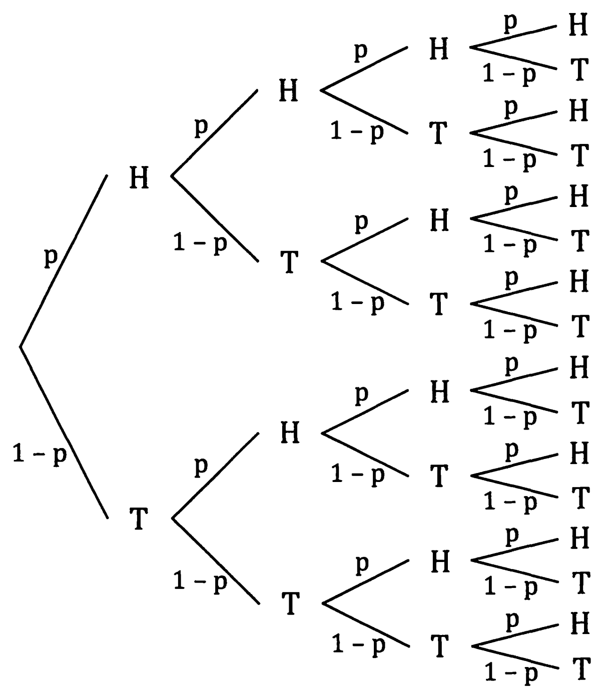
☞ A Bernoulli random variable can be used as the probability model for situations involving two mutually exclusive outcomes.

Binomial distributions.

If a Bernoulli trial is performed repeatedly, with the probability of success in a trial occurring with constant probability, i.e. the trials are **independent**, the distribution that arises by considering the number of successes is called a **binomial distribution**.

For example, suppose that a biased coin is flipped four times and that on each flip the probability of the result being a head, which we will call a success, is p. Each flip of the coin is a Bernoulli trial with P(success) = p and P(failure) = 1 – p.

The probability tree diagram for this coin flipping is as follows:



$P(HHHH) = p^4$
$P(HHHT) = p^3\,(1-p)$
$P(HHTH) = p^3\,(1-p)$
$P(HHTT) = p^2\,(1-p)^2$
$P(HTHH) = p^3\,(1-p)$
$P(HTHT) = p^2\,(1-p)^2$
$P(HTTH) = p^2\,(1-p)^2$
$P(HTTT) = p\,(1-p)^3$
$P(THHH) = p^3\,(1-p)$
$P(THHT) = p^2\,(1-p)^2$
$P(THTH) = p^2\,(1-p)^2$
$P(THTT) = p\,(1-p)^3$
$P(TTHH) = p^2\,(1-p)^2$
$P(TTHT) = p\,(1-p)^3$
$P(TTTH) = p\,(1-p)^3$
$P(TTTT) = (1-p)^4$

Notice that "2 heads and 2 tails in any order" occur on 6 of the final outcomes, and each with probability $p^2\,(1-p)^2$.

If X is the number of heads this procedure produces, then

$$P(X = 2) = 6\,p^2\,(1-p)^2$$

The complete probability distribution for X is as follows:

x	0	1	2	3	4
$P(X=x)$	$(1-p)^4$	$4\,p\,(1-p)^3$	$6\,p^2\,(1-p)^2$	$4\,p^3\,(1-p)$	p^4

Suppose instead that this coin were flipped eight times and we again use X for the number of heads. What would $P(X = 2)$ be now?

The tree diagram would be large and tedious to construct so instead we "think it through" as explained on the next page.

From the start of such a tree diagram, to a final outcome involving "2 heads and 6 tails" we travel along eight branches, 2 of which have probability p, and the other 6 having probability (1 – p).

Question: How many such branches are there?

Answer: The same number as there are ways of choosing which 2 of the 8 branches will be the "p branches", i.e. ${}^{8}C_2$ (= 28).

Thus
$$P(X=2) = {}^{8}C_2\, p^2 (1-p)^6 = 28\, p^2 (1-p)^6$$

Similarly
$$P(X=3) = {}^{8}C_3\, p^3 (1-p)^5 = 56\, p^3 (1-p)^5$$

$$P(X=4) = {}^{8}C_4\, p^4 (1-p)^4 = 70\, p^4 (1-p)^4 \qquad \text{etc.}$$

To generalise:

Suppose that an event or trial is repeated n times, and in each trial we consider only two outcomes, A and A'. Further suppose that in each trial the probability of event A occurring is p (from which it follows that P(A') = 1 – p).

In the n trials, the probability of A occurring x times ($x \le n$), is:

$$\boxed{{}^{n}C_x\, p^x (1-p)^{n-x}}$$

or using the form $\binom{n}{x}$ for ${}^{n}C_x$:

$$\boxed{\binom{n}{x} p^x (1-p)^{n-x}}$$

Example 1

A normal fair die is rolled ten times.

Determine the probability of obtaining exactly (a) three sixes (b) five sixes.

If the discrete random variable X is the number of sixes in ten rolls of the die then

$$P(X=x) = {}^{10}C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{10-x}$$

(a) $P(X=3) = {}^{10}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 \approx 0{\cdot}155$

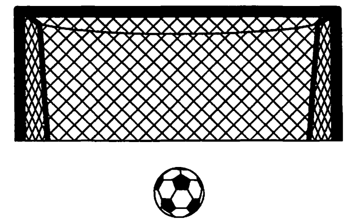
In 10 rolls of a normal die the probability of obtaining exactly three sixes is approximately 0·155.

(b) $P(X=5) = {}^{10}C_5 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^5 \approx 0{\cdot}013$

In 10 rolls of a normal die the probability of obtaining exactly five sixes is approximately 0·013.

Example 2

Suppose that each time a particular soccer player takes a penalty kick the probability of scoring a goal is 0·6. Determine the probability that if the player takes eight penalty kicks they will score (a) exactly five times,
(b) at least five times.



(a) Defining the random variable X as the number of goals scored in the eight attempts, X can take the values 0, 1, 2, 3, 4, 5, 6, 7, 8.

$P(X = x) = {}^8C_x\,(0{\cdot}6)^x\,(0{\cdot}4)^{8-x}$ Thus $P(X = 5) = {}^8C_5\,(0{\cdot}6)^5\,(0{\cdot}4)^3 \approx 0{\cdot}279$

The probability of the player scoring exactly five goals in eight penalty attempts is approximately 0·28.

(b)
$$\begin{aligned}
P(X \geq 5) &= P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)\\
&= {}^8C_5(0{\cdot}6)^5(0{\cdot}4)^3 + {}^8C_6(0{\cdot}6)^6(0{\cdot}4)^2 + {}^8C_7(0{\cdot}6)^7(0{\cdot}4)^1 + {}^8C_8(0{\cdot}6)^8(0{\cdot}4)^0\\
&\approx 0{\cdot}2787 + 0{\cdot}2090 + 0{\cdot}0896 + 0{\cdot}0168\\
&= 0{\cdot}594 \text{ correct to three decimal places.}
\end{aligned}$$

The probability of the player scoring at least five goals in eight penalty attempts is approximately 0·59.

Note

- If a Bernoulli trial is performed n times, and the probability of success in each trial is p, the probability of exactly x successes in the n trials is
$${}^nC_x\,p^x\,(1 - p)^{n-x}$$
- The number of trials, n, and the probability of success on each trial, p, are called the **parameters** of the distribution. If we know that a random variable is binomially distributed and the parameters n and p are known, the probability distribution can be completely determined.
- If the discrete random variable X is binomially distributed with parameters n and p this is sometimes written as:
$$X \sim \text{b}(n, p), \quad X \sim \text{B}(n, p), \quad X \sim \text{bin}(n, p) \quad \text{or} \quad X \sim \text{Bin}(n, p).$$
- For a binomial distribution involving n trials, with p the probability of success on each trial:

Mean, i.e. E(X), = np

and Var(X) = np(1 − p) or npq, where $q = 1 - p$, the probability of failure on each trial.

Hence **SD(X) = $\sqrt{\mathbf{np(1-p)}}$** or $\sqrt{npq}$ where $q = (1 - p)$.

- Considering the number of marbles obtained of a certain colour when marbles are selected from a bag, with each marble replaced before the next is drawn, will involve a binomial distribution. If replacement does not occur the probability of "success" is not the same for each trial and the distribution would not be binomial. However if the bag contains a very large number of marbles, and the sample size is comparatively small, the probability of success is almost constant and the binomial distribution could be used to model the situation. This is often the situation in an opinion poll when a small sample is chosen from a large population.

Example 3

A Bernoulli trial has P(success) = p and P(failure) = (1 – p). The trial is carried out five times, with each trial outcome independent of the outcomes of the other trials. If the random variable X is the number of successes achieved in these five trials show the probability distribution of X in terms of p.

In this case $X \sim \text{Bin}(5, p)$ and so: $P(X = x) = \binom{5}{x} p^x (1 - p)^{5-x}$

Thus $P(X = 0) = \binom{5}{0} p^0 (1 - p)^5$ $\quad P(X = 1) = \binom{5}{1} p^1 (1 - p)^4$

$P(X = 2) = \binom{5}{2} p^2 (1 - p)^3$ $\quad P(X = 3) = \binom{5}{3} p^3 (1 - p)^2$

$P(X = 4) = \binom{5}{4} p^4 (1 - p)^1$ $\quad P(X = 5) = \binom{5}{5} p^5 (1 - p)^0$

The complete distribution would be:

x	0	1	2	3	4	5
$P(X = x)$	$(1 - p)^5$	$5p(1 - p)^4$	$10p^2(1 - p)^3$	$10p^3(1 - p)^2$	$5p^4(1 - p)$	p^5

Notice that if we write (1 – p) as q, this becomes

x	0	1	2	3	4	5
$P(X = x)$	q^5	$5pq^4$	$10p^2q^3$	$10p^3q^2$	$5p^4q$	p^5

the same terms as are obtained in the expansion of $(q + p)^5$:

$$(q + p)^5 = q^5 + 5pq^4 + 10p^2q^3 + 10p^3q^2 + 5p^4q + p^5$$

Why should this be?

Well, when determining $P(X = 2)$, the ${}^5C_2$ ways (= 10) arises when we consider the number of ways the two "p branches" could be chosen from the five branches. In the expansion of $(q + p)^5$, the p^2q^3 term involves with the number of ways the two brackets supplying the p could be chosen from the five brackets. Once again ${}^5C_2$ is involved.

Example 4
Each question of a multiple choice test paper offers five answers, one of which is correct. A student answers 7 questions by simply guessing which response is correct each time. If we define the random variable X as how many of these seven questions the student gets correct determine the probability distribution for X, giving probabilities correct to 3 decimal places.

N$^{o.}$ of trials = 7. P(success, i.e. gets question correct) = 0·2. $X \sim \text{Bin}(7, 0{\cdot}2)$.

$$P(X=0) = {}^7C_0\, 0{\cdot}2^0\, 0{\cdot}8^7 \approx 0{\cdot}2097 \qquad P(X=1) = {}^7C_1\, 0{\cdot}2^1\, 0{\cdot}8^6 \approx 0{\cdot}3670 \qquad P(X=2) = {}^7C_2\, 0{\cdot}2^2\, 0{\cdot}8^5 \approx 0{\cdot}2753$$

$$P(X=3) = {}^7C_3\, 0{\cdot}2^3\, 0{\cdot}8^4 \approx 0{\cdot}1147 \qquad P(X=4) = {}^7C_4\, 0{\cdot}2^4\, 0{\cdot}8^3 \approx 0{\cdot}0287 \qquad P(X=5) = {}^7C_5\, 0{\cdot}2^5\, 0{\cdot}8^2 \approx 0{\cdot}0043$$

$$P(X=6) = {}^7C_6\, 0{\cdot}2^6\, 0{\cdot}8^1 \approx 0{\cdot}0004 \qquad P(X=7) = {}^7C_7\, 0{\cdot}2^7\, 0{\cdot}8^0 \approx 0{\cdot}0000$$

The complete probability distribution is, correct to three decimal places:

x	0	1	2	3	4	5	6	7
$P(X=x)$	0·210	0·367	0·275	0·115	0·029	0·004	0·000	0·000

Example 5
When driving to work a motorist encounters 8 sets of traffic lights.
Let us suppose that for each of these the probability that the motorist has to stop at the lights is a constant 0·4. Find the probability that in the journey to work the motorist has to stop at (a) exactly six of the eight sets of lights,
(b) at least six of the eight sets of lights.

If X is the number of lights the motorist stops at then $X \sim \text{Bin}(8, 0{\cdot}4)$

(a)
$$\begin{aligned} \text{P(stop at exactly six)} &= P(X=6) \\ &= {}^8C_6\, 0{\cdot}4^6\, 0{\cdot}6^2 \\ &\approx 0{\cdot}041 \end{aligned}$$

(b)
$$\begin{aligned} \text{P(stop at at least six)} &= \text{P(stop at 6)} + \text{P(stop at 7)} + \text{P(stop at 8)} \\ &= P(X=6) + P(X=7) + P(X=8) \\ &= {}^8C_6\, 0{\cdot}4^6\, 0{\cdot}6^2 + {}^8C_7\, 0{\cdot}4^7\, 0{\cdot}6^1 + {}^8C_8\, 0{\cdot}4^8\, 0{\cdot}6^0 \\ &\approx 0{\cdot}050 \end{aligned}$$

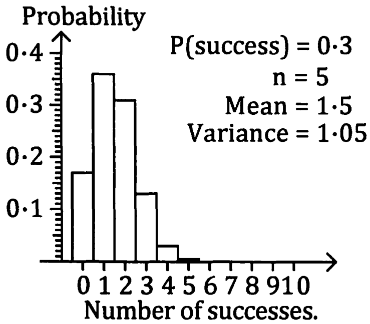
Graphs of binomial distributions.

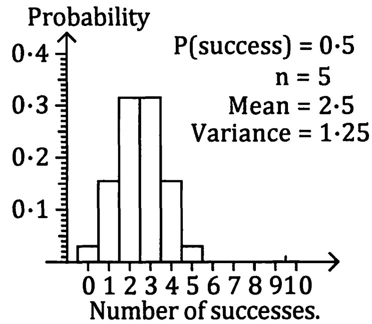
The graphs for the binomial distributions with

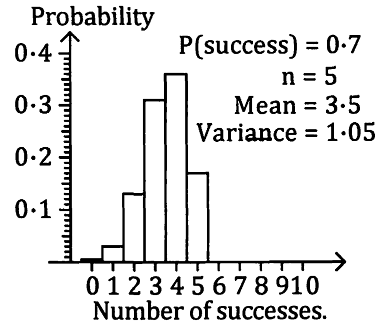
$$p = 0{\cdot}3, \quad p = 0{\cdot}5 \quad \text{and} \quad p = 0{\cdot}7$$

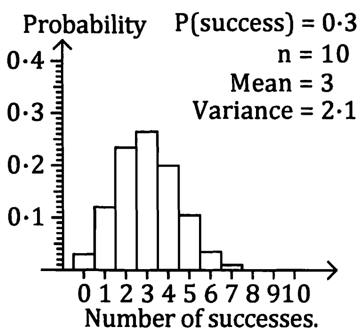
are shown below for

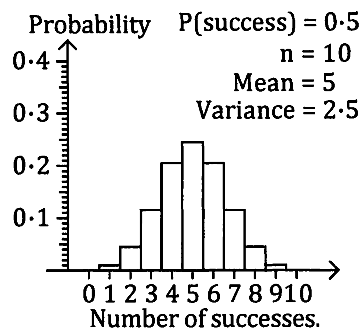
$$n = 5, \quad n = 10 \quad \text{and} \quad n = 15$$

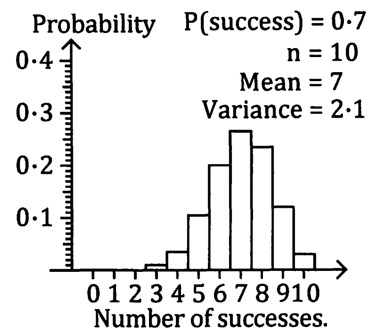


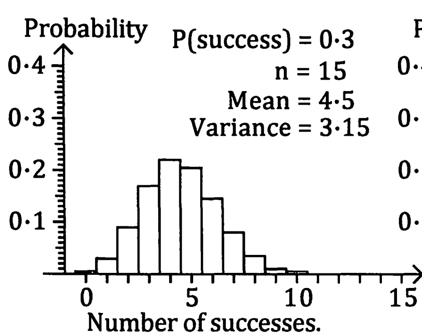


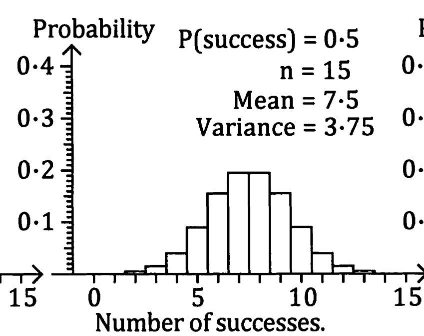


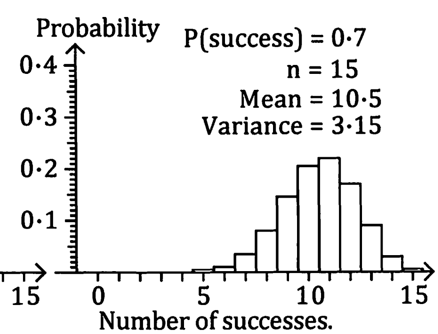












Note
- the symmetrical nature of the graphs for which P(success) = 0·5.
- the skewed nature of the graphs for which P(success) ≠ 0·5.
- the graphs of P(success) = k and P(success) = 1 – k are mirror images.
- the graphs for P(success) ≠ 0·5 appear to move towards a more symmetrical distribution as n increases.

Activity

Get your calculator or computer spreadsheet to output random numbers between 0·00000 and 0·99999 and record the first five digits after the decimal point.

For example, for:	0·77641	0·21391	0·35890	0·53652	0·79146	0·49946
record:	77641	21391	35890	53652	79146	49946

Note how many digits in each group of five are 1s, 2s or 3s:

7764$\underline{1}$	$\underline{2}\underline{1}\underline{3}9\underline{1}$	$\underline{3}$5890	5$\underline{3}$65$\underline{2}$	79$\underline{1}$46	49946
1	4	1	2	1	0

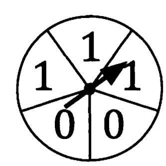
Do this for one hundred sets of five digits and tabulate your results:

N$^{o.}$ of 1s, 2s and 3s in set of 5	0	1	2	3	4	5
Tally	/	///	/		/	
Frequency						

Compare your results with those suggested by modelling this activity using a binomial distribution with n = 5

and P(success) = P(digit being 1, 2 or 3)
= 0·3.

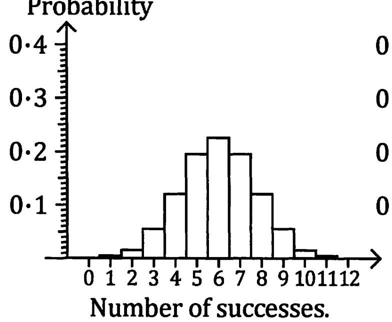
Exercise 9A



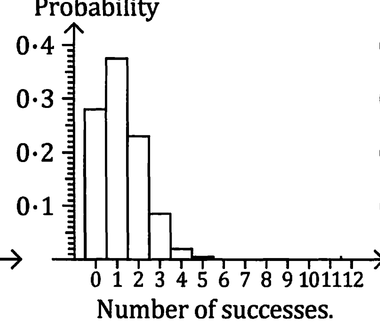
1. Find the mean, or expected value, and the variance of a Bernoulli distribution with parameter 0·6.

2. The three graphs below show binomial distributions for
   n = 12 and P(success) = 0·1, 0·5 and 0·8.
   Which graph has which P(success) value?

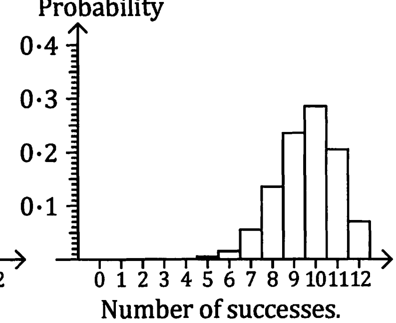
Graph A



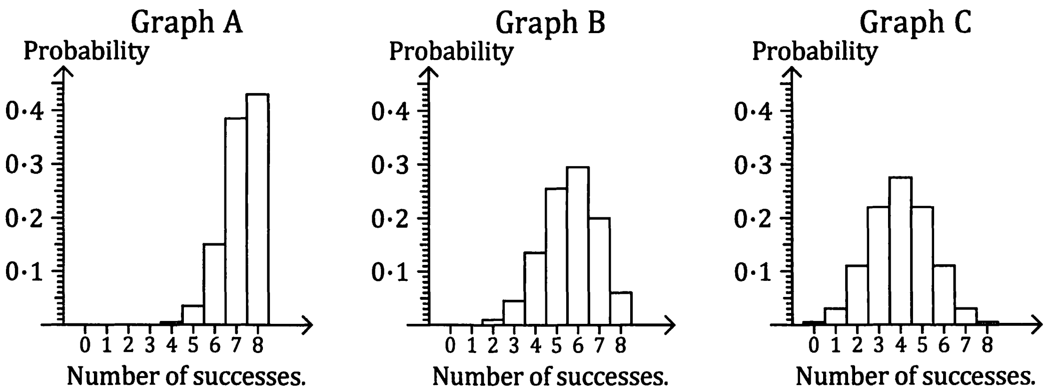
Graph B



Graph C



3. The three graphs below show binomial distributions for
$n = 8$ and P(success) = 0·5, 0·7 and 0·9.
Which graph has which P(success) value?



4. Find the mean and standard deviation of a binomial distribution with n, the number of trials, equal to 12 and p, the probability of success in each trial equal to 0·25.

5. A binomial distribution has a mean of 9·6 and a standard deviation of 2·4. Find n, the number of trials and p, the probability of success in each trial.

6. The discrete random variable X is binomially distributed with parameters $n = 8$ and $p = 0{\cdot}25$. The probability distribution for X is shown below:

x	0	1	2	3	4	5	6	7	8
$P(X = x)$ Rounded to 4dp.	0·1001	0·2670	0·3115	a	b	0·0231	0·0038	0·0004	0·0000

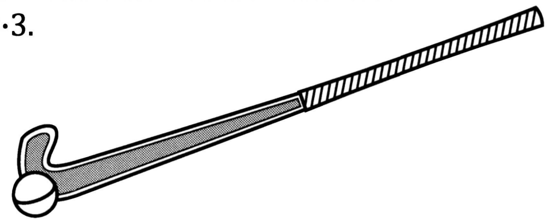
(a) Find the values of a and b.
(b) Find the mean, μ, and the standard deviation, σ, of X.
(c) Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$ giving your answer rounded to 3 dp.

7. The discrete random variable X is binomially distributed with parameters $n = 9$ and $p = 0{\cdot}6$.
Find (a) $P(X = 8)$ (b) $P(X = 9)$ (c) $P(X \geq 8)$ (d) $P(X < 8)$

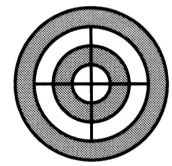
8. If $X \sim \text{Bin}(6, 0{\cdot}7)$ determine:
(a) $P(X = 5)$ (b) $P(X = 6)$ (c) $P(X \geq 5)$ (d) $P(X < 5)$

9. A normal fair die is rolled eight times.
Determine the probability of obtaining exactly (a) two sixes (b) six sixes.

10. The coach of a hockey team feels that for each short corner the team takes the probability of a goal resulting is 0·3.
Assuming this probability is correct what is the probability that when this team takes nine short corners, four goals will result.



11. Suppose that each time I shoot at a target the probability of my shot scoring a "bull" is 0·7.
Assuming that this probability remains constant, determine the probability that in ten such shots I will score
(a) 6 bulls (b) 8 bulls (c) more than 8 bulls (d) at least 8 bulls.



12. Each question of a multiple choice test paper offers four answers, one of which is correct. A student answers 20 questions by randomly guessing which response is correct each time. If we define the random variable X as how many of these twenty questions the student gets correct determine, correct to 3 decimal places
(a) $P(X = 5)$ (b) $P(X = 10)$ (c) $P(8 < X \leq 10)$

13. A particular disease is fatal if not treated. However the treatment itself is particularly risky. For anyone with the disease and given the treatment the probability that the treatment will work is 0·4. If six people with the disease are given the treatment what is the probability that the treatment will work for more than half of these six?

Values from tables and calculators.

Nowadays individual probabilities and cumulative probabilities for various distributions can be obtained directly from many calculators without having to input complicated expressions.

For example, the display on the right shows that for $X \sim \text{Bin}(4, 0{\cdot}3)$,

$$P(X = 2) = 0{\cdot}2646$$

and $$P(X \leq 2) = 0{\cdot}9163$$

(The letters PDf in the display stands for *probability density function*. CDf indicates *cumulative* probabilities are being given.)

```
binomialPDf(2,4,0.3)
                          0.2646
binomialCDf(0,2,4,0.3)
                          0.9163
```

Some calculators are also able to display the graph of the distribution as well.

Not all calculators require the same order of input suggested by the display shown above. Get to know your calculator with regard to its ability to display values from binomial probability distributions.

Prior to the ready availability of such calculators binomial probabilities were obtained either by use of the formula, as we did in the previous exercise, or from books displaying tables of probabilities. Part of a typical table display of binomial probabilities is shown below. (Other pages of the book would show cumulative probabilities.)

Can you use the table shown below to find the probability given on the previous page, i.e. that for $X \sim \text{Bin}(4, 0{\cdot}3)$,

$$P(X = 2) = 0{\cdot}2646\ ?$$

n	x	p										
		0·1	0·2	0·25	0·3	0·4	0·5	0·6	0·7	0·75	0·8	0·9
2	0	·8100	·6400	·5625	·4900	·3600	·2500	·1600	·0900	·0625	·0400	·0100
	1	·1800	·3200	·3750	·4200	·4800	·5000	·4800	·4200	·3750	·3200	·1800
	2	·0100	·0400	·0625	·0900	·1600	·2500	·3600	·4900	·5625	·6400	·8100
3	0	·7290	·5120	·4219	·3430	·2160	·1250	·0640	·0270	·0156	·0080	·0010
	1	·2430	·3840	·4219	·4410	·4320	·3750	·2880	·1890	·1406	·0960	·0270
	2	·0270	·0960	·1406	·1890	·2880	·3750	·4320	·4410	·4219	·3840	·2430
	3	·0010	·0080	·0156	·0270	·0640	·1250	·2160	·3430	·4219	·5120	·7290
4	0	·6561	·4096	·3164	·2401	·1296	·0625	·0256	·0081	·0039	·0016	·0001
	1	·2916	·4096	·4219	·4116	·3456	·2500	·1536	·0756	·0469	·0256	·0036
	2	·0486	·1536	·2109	·2646	·3456	·3750	·3456	·2646	·2109	·1536	·0486
	3	·0036	·0256	·0469	·0756	·1536	·2500	·3456	·4116	·4219	·4096	·2916
	4	·0001	·0016	·0039	·0081	·0256	·0625	·1296	·2401	·3164	·4096	·6561

Example 6

Given that $X \sim \text{Bin}(10, 0{\cdot}1)$, find (a) $P(X = 3)$ (b) $P(X \leq 3)$ (c) $P(X = 3 | X \leq 3)$

Using tables, or a calculator:

(a) $P(X = 3) = 0{\cdot}0574$

(b) $P(X \leq 3) = 0{\cdot}9872$

(c) $P(X = 3 | X \leq 3) = \dfrac{0{\cdot}0574}{0{\cdot}9872}$

$\approx 0{\cdot}058$

```
binomPdf(10,0.1,3)
                    0.057396
binomCdf(10,0.1,0,3)
                    0.987205
```

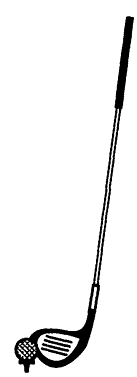
Assessing improvement using a binomial model.

To test the effectiveness of their two hour

"Improve Your Golf Swing"

course the organisers arrange for ten attendees to play a shot before attending the course and then play a similar shot after the course. For each shot the organisers measured how far each person's shot finished from the target flag.

Suppose eight of the ten finished closer to the flag with their second attempt, i.e. after attending the course, than before it. Could we conclude that their attendance at the course was responsible for the apparent improvement?



One aspect we could investigate (as well as considering such things as just how similar the two shots were, for example, if the weather conditions were the same) would be to see how likely it is that eight, or more, of the ten would improve purely by chance, and not by attending the course.

Let us suppose that the probability of someone attempting this shot and improving on their second attempt "by pure luck", and not by attending the course, is 0·5. The chance that out of ten people, eight or more will improve at the second attempt is then $P(X \geq 8)$ with $X \sim \text{Bin}(10, 0{\cdot}5)$.

In this case $P(X \geq 8) = 0{\cdot}0547$.

Thus there is only a 5 or 6 percent probability that eight or more of the golfers would improve "by chance" (if our binomial model and the 0·5 probability is appropriate).

Hence the fact that eight did improve suggests that the improvement could be for some reason, rather than pure chance, and that could be their attendance at the course. However further investigation would be advisable before making too many claims about the ability of the course to bring such improvement.

Exercise 9B

1. The discrete random variable X is binomially distributed with parameters n = 8 and p = 0·2. Use your calculator or a book of tables to determine:
   (a) $P(X = 4)$ (b) $P(X = 6)$ (c) $P(X \leq 6)$ (d) $P(X < 7)$

2. The discrete random variable X has a binomial distribution with parameters n = 20 and p = 0·6. Use your calculator or a book of tables to determine:
   (a) $P(X = 10)$ (b) $P(X = 14)$ (c) $P(X \leq 14)$ (d) $P(X < 15)$

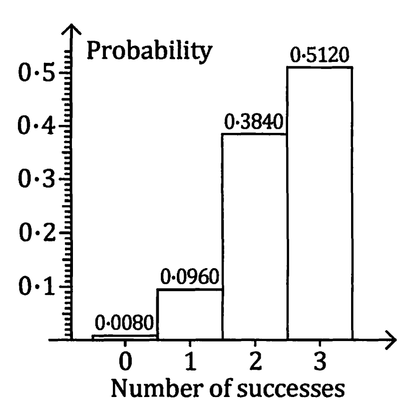
3. Given that $X \sim \text{Bin}(9, 0{\cdot}4)$, find:
   (a) $P(X = 2)$ (b) $P(X \leq 3)$ (c) $P(X = 2 | X \leq 3)$

4. Given that $X \sim \text{Bin}(20, 0{\cdot}7)$, find:
   (a) $P(X = 15)$ (b) $P(X \geq 15)$ (c) $P(X = 15 | X \geq 15)$

5. Given that $X \sim \text{Bin}(15, 0{\cdot}8)$, find:
   (a) $P(X \geq 7 | X \leq 10)$ (b) $P(X \leq 10 | X \geq 7)$

6. Given that $X \sim \text{Bin}(12, 0{\cdot}3)$, find:
   (a) $P(X \leq 5 | X \geq 3)$ (b) $P(X \geq 3 | X \leq 5)$

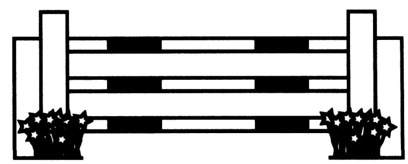
7. The graph on the right shows a binomial probability distribution.
   Use the table on the previous page to determine the probability of success on each of the Bernoulli trials involved in this distribution.



8. A biased coin is such that on each flip the probability of getting a head is 0·4. The coin is flipped 20 times. Find the probability of obtaining
(a) exactly 12 heads, (b) no more than 12 heads, (c) at least 12 heads.

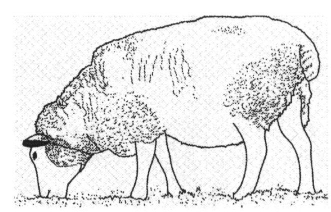
9. A gardener sells punnets containing eight seedlings. In an attempt to minimise the number of punnets that cannot be sold because they contain less than eight seedlings, the gardener plants ten seeds in each punnet and, if more than eight germinate he takes the extra ones out. If each seed has a probability of 0·9 of germinating what is the probability that of the ten seeds placed in a punnet
(a) exactly eight will germinate, (b) at least eight will germinate,
(c) less than eight will germinate?

10. At each of the fifteen fences in an equestrian cross country event, Kerry can either go clear or incur penalty points. If the probability of her incurring penalty points at any fence is a constant 0·1, find the probability that in the fifteen fence event she incurs penalty points at
(a) exactly three of the fifteen fences,
(b) less than three of the fifteen fences,
(c) more than three of the fifteen fences.



11. Each question of a multiple choice test paper offers five answers, one of which is correct. A student answers 20 questions by randomly guessing which response is correct each time. If we define X as the number of questions the student gets correct determine, correct to 3 decimal places
(a) $P(X = 5)$, (b) $P(X = 10)$, (c) $P(X \geq 10)$, (d) $P(3 \leq X \leq 7)$.

12. Suppose the probability of a particular hereditary characteristic being passed from a ewe to each lamb she gives birth to is 0·25. If, over a period of time, the ewe gives birth to six lambs determine, correct to four decimal places, the probability that
(a) none of the six will have the characteristic,
(b) all six will have the characteristic,
(c) exactly three will have the characteristic,
(d) at least three will have the characteristic.



13. The probability of any randomly chosen component being faulty is 0·01.
If ten of these components are randomly selected what is the probability that at least one will be defective?

14. Two fair dice are rolled and the two numbers on the uppermost faces are added together.
(a) In one roll of these dice what is the probability of obtaining a total of 7?
If these dice are rolled ten times and the total obtained is noted each time, what is the probability of obtaining
(b) a total of seven at least once,
(c) a total of seven on less than three of the ten occasions,
(d) a total of seven on at least three of the ten occasions?

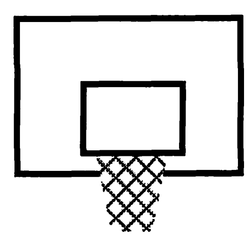
15. A multiple choice test contains 12 questions each offering 4 answers, only one of which is correct each time. A student knows the correct answers to 7 of the questions but randomly guesses the answers to the remaining 5. What is the probability that the student will get at least 10 out of the 12 correct?

16. Matt and Joel play soccer and both are strikers. Matt tends to have more shots on goal during a match than Joel does but, for each shot on goal, Joel seems to have a better chance of scoring.

Let us suppose that for each shot on goal that Matt has, the probability of the shot scoring is 0·2, and for each shot on goal that Joel has, the probability of the shot scoring is 0·4.
If Joel has three attempts on goal and Matt has six, which of these two strikers has the greater probability of scoring at least one goal?

17. If I flip a fair coin twice then P(≤ 2 heads) = 1.
If I flip a fair coin three times P(≤ 2 heads) = 0·875.
If I flip a fair coin four times P(≤ 2 heads) = 0·6875.
What is the greatest number of times I can flip a fair coin and still have the value of P(≤ 2 heads) above 0·2?

18. Let us suppose that each time a particular basketball player shoots a three point attempt the probability of scoring is 0·4. How many three point attempts does this player need to make for the probability of at least three successes to exceed 0·75?

19. Prior to the arrival of a specialist shooting coach for a weekend workshop, the twenty members of a basketball club took part in a shooting drill in which they took one shot at basket from each of fifteen places on a basketball court. The number of successes out of fifteen was recorded for each player.

At the end of the weekend workshop the twenty members again took one shot from each of the same fifteen places and scores out of fifteen were again recorded.
Sixteen of the twenty members recorded higher scores in the "after the workshop test" than the "before the workshop" test.
Comment on this improvement.

20. Fifty students sat a multiple choice test which involved fifteen questions.
On each question the students had to select one of four answers a, b, c or d.
After the test the students complained that whilst two of the questions involved work they had been taught, the other thirteen were on things they had been given no prior information about and they were left having to guess answers for those.
The scores the students obtained were as follows:

Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Nº· of students	0	0	0	6	9	16	8	4	5	0	1	0	1	0	0	0

Do you believe the complaint made by the students? (Justify your answer.)

Miscellaneous Exercise Nine.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. An object with a temperature of 80°C is placed in an environment with temperature 15°C. The temperature of the object, t seconds later, is T°C, where T approximately follows the mathematical rule
$$T = 15 + 65e^{-0 \cdot 004t}.$$
Find the temperature of the object after (a) 5 minutes, (b) 10 minutes.

Differentiate each of the following with respect to x.

2. $(x+3)(x+2)$	3. $(2x+1)^3$	4. $(3-2x)^2$
5. $\dfrac{x}{x+1}$	6. $(2x+1)(6x^3-5)$	7. $x(2-3x)^3$
8. $(2x+1)(4+7x)^4$	9. e^x	10. $5x^2+e^x$
11. $4e^{3x}+x^4-2$	12. e^{2x-4}	13. e^{3x+1}
14. x^2e^x	15. $x+xe^x$	16. $\sin x$
17. $\cos 3x$	18. $\sin(3x-5)$	19. $e^{2x}\sin 4x$

20. Given that $X \sim \text{Bin}(20, 0\cdot25)$, find $P(X = 12)$. (Round your answer to 6 dp.)

21. We are interested in the number of successes a person is likely to have when they attempt to guess the outcome of a normal fair die being rolled eight times. Explain why a binomial distribution would be a suitable model for this situation.

22. Differentiate each of the following with respect to x without the assistance of your calculator, and then use your calculator to check your answers.

(a) $\dfrac{1}{\sqrt{3x-1}}$ (b) $\dfrac{5x^2+1}{x}$ (c) $\displaystyle\int_3^x \frac{t-1}{t^3}\,dt$ (d) $\displaystyle\int_1^x \frac{1+t^3}{\sqrt{t}}\,dt$

23. If $f(x) = 2(2x-1)^3$ find (a) $f(2)$, (b) $f(0\cdot5)$, (c) $f'(x)$, (d) $f'(3)$.

24. Find the exact gradient of $y = 3x^2 + e^{2x} + 3$ at the point $(1, 6 + e^2)$.

25. Integrate the following with respect to x without the assistance of your calculator, and then use your calculator to check your answers (but remember that your answers should include "+ c").

(a) $15x^4$	(b) $6x^2 - 4x + 6$	(c) $\dfrac{x+3}{\sqrt{x}}$
(d) $(2x+3)^5$	(e) $(5x-2)^3$	(f) $\sin x$
(g) $\cos 2x$	(h) $\sin(2x-1)$	(i) $4\sin 3x$
(j) $4x(x^2+3)^4$	(k) $\dfrac{d}{dx}(x^5-7x)$	(l) $\dfrac{d}{dx}(e^x\sqrt{x}-7x)$

26. If $A = \dfrac{6x+3}{x-1}$ find an expression for the rate of change of A with respect to x.

27. If $T = 3\sqrt{p}$ find the rate of change of T with respect to p when
(a) $p = 16$, (b) $p = 25$, (c) $p = 36$.

28. Determine the gradient of the curve $y = (2x+3)(x^2+3)$ at the point $(-1, 4)$.

29. Find y in terms of x given that $\dfrac{dy}{dx} = 2x - 5$ and when $x = 1$, $y = 7$.

30. Find y as a function of x given that $\dfrac{dy}{dx} = 10x - 6$ and $y = 9$ when $x = 2$.

31. Find $f(x)$ given that $f'(x) = 12(8 - 2x)^2$ and $f(4) = 6$.

32. Find x in terms of t given that $\dfrac{dx}{dt} = -\dfrac{18}{(3t+1)^2}$ and $x = 4{\cdot}5$ when $t = 1$.

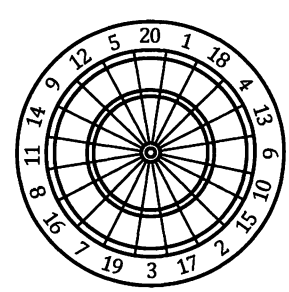
33. If $\dfrac{dy}{dx} = 5(2x+1)^4$ and $y = 125$ when $x = 1$ find
(a) y in terms of x,
(b) y, when $x = 0$,
(c) x, when $y = 19{\cdot}5$, $(x \in \mathbb{R})$.

34. Use calculus to determine the area between $y = 2x + 1$ and the x-axis from $x = -2$ to $x = 2$. Check your answer using area formulae.

35. Find y as a function of x given that $\dfrac{d^2y}{dx^2} = 30x - 14$, $y = 1$ when $x = 1$
and $y = -9$ when $x = -1$.

36. If \$500 is invested at 12% per annum compounded continuously, the account grows to $\$500e^{0{\cdot}12t}$ after t years. What is the instantaneous rate of growth, in dollars per year correct to two decimal places, when
(a) $t = 1$, (b) $t = 5$, (c) $t = 10$, (d) $t = 25$?

37. Manuel estimates that each time he throws a dart at the dartboard, aiming at treble 20, the chance of his dart successfully scoring treble 20 is 0·1. What is the probability that in ten such attempts he will successfully score treble 20 at least once? (Give your answer rounded to two decimal places.)



38. If $f(x) = 3x^2 + x$ use differentiation to find the approximate change in the value of the function when x changes from 5 to 5·04.
Compare your answer to $f(5{\cdot}04) - f(5)$.

39. If $V = 5x^3$ use differentiation to find the approximate percentage change in V when x changes by 3%.

40. Find the area enclosed between $y = 2\sin x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

41. An engineer requires a function of the form $f(x) = \frac{2x+3}{2x+a}$, for constant a, to be such that $f'(3) = -16$.
With the assistance of your calculator if you wish, find the possible value(s) of a.

42. A charitable organisation sets up a fundraising craft stall in the centre of a city, selling items made by members. The council allows the group to operate the stall from 9 a.m. to midday one Saturday morning. The group finds that the amount in their cash box grows during this time from the original amount they started with as a cash float to $\$A$ where $\frac{dA}{dt} = 1{\cdot}2e^{0{\cdot}01t}$, t being the number of minutes past 9 a.m. $(0 \le t \le 180)$.
How much did they raise during the three hours (to the nearest dollar)?

43. An object moves such that its displacement, x metres, from an origin, O, at time t seconds is given by $x = e^{\cos t}$.
Find an expression for the velocity of the object at time t seconds and determine the velocity of the object when $t = \pi/2$.

44. A fair six sided die has its faces numbered 1, 1, 3, 3, 3, 6.
The die is thrown twice and the number on the uppermost face is noted each time and the two numbers are then added together.
Find (a) the probability that the total obtained is 6,
(b) the expected mean value of the total if this activity were to be carried out many times.

45. If $y = \frac{1+\sin x}{1-\sin x}$ determine $\frac{dy}{dx}$.

46. A normal fair six sided die is rolled 4 times.
In theory this could result in 0, 1, 2, 3 or 4 sixes.
(a) Which of these numbers is the most likely number of sixes to occur?
(b) Is it more likely that this number of sixes will result or more likely that it will not result?
(c) If this "4 roll event" was repeated many times what would you expect the long term average number of sixes obtained per event to be?

47. Determine the exact gradient of $y = \frac{\cos x}{x}$ at the point $\left(\frac{\pi}{3}, \frac{3}{2\pi}\right)$.

48. Use calculus to determine the nature and exact coordinates of any turning points on the curve $y = \sqrt{3}\sin x + \cos x$ for $-2\pi < x < 2\pi$.

49. A uniform discrete random variable, X, can take the integer values 1, 2, 3, 4, 5, ..., n.
Find E(X) in terms of n.
Remember: The sum to n terms of the arithmetic progression with first term a and common difference d is $\frac{n}{2}(2a + (n-1)d)$.

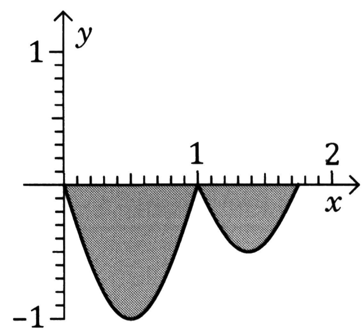
50. The point $(-3, a)$ lies on the curve $y = \dfrac{5x-7}{2x+10}$ and the tangent to the curve at this point is parallel to $y = bx + 3$, and cuts the y-axis at $(0, c)$.
Determine (a) the values of the constants a, b and c,
(b) the coordinates of the other point on the curve where the tangent is parallel to $y = bx + 3$.

51. A yacht designer is investigating possible shapes for fins attached to the keel of a yacht. One possibility involves parts of the following curves and is shown on the right.

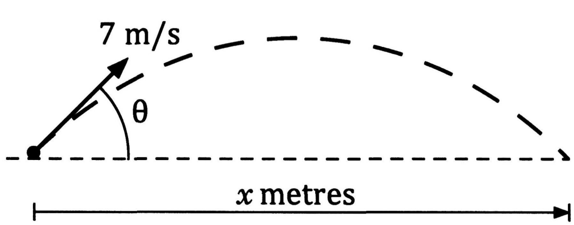
$$y = -\sin(\pi x)$$

$$y = -\frac{1}{2}\sin\left(\frac{4\pi}{3}(x-1)\right)$$

Find the total area shaded.



52. If an object is projected from a point on horizontal ground, with an initial speed of 7 m/s and at an angle of θ to the horizontal, the distance from the point of projection to the point of landing is x metres, where $x = 10\sin\theta\cos\theta$.



(a) Use the product rule to find an expression for $\frac{dx}{d\theta}$, for θ in radians.

(b) Use calculus to show that for x to be a maximum the angle of projection needs to be 45° to the horizontal.
Justify that this would indeed give a maximum value for x (as opposed to a minimum or horizontal inflection) and find this maximum value.

(c) Given the trigonometric identity:

$$2\sin\theta\cos\theta = \sin 2\theta$$

express x in terms of $\sin 2\theta$ and then use your knowledge of the amplitude of sin A, and the value of A for which it occurs, to confirm your answers to part (b).